In the name of the Almighty Allah

Final exam - Real Analysis 1 *The Amirkabir University* 14/10/89 Time: 3 hrs.

Over all the following questions μ and ϕ are positive measures on a σ-algebra $\mathfrak{M}$ in a set X.

1. Show that if $\mathfrak{M}$ is infinite, then $\mathfrak{M}$ is uncountable.

2. If $a_{ij} \geq 0$ for $i, j = 1, 2, \ldots$, apply the Fubini's Theorem 8.8 to deduce

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

3. Let X be a topological space and $\{\psi_\alpha, \alpha \in A\}$ be a family of lower semicontinuous functions on X. Show that the function $g(x) = \sup_{\alpha \in A} \psi_\alpha(x)$ is also lower semicontinuous.

4. For $0 < p < 1$ define $d(f, g) = \int_X |f|^p d\mu$. Prove that d is a metric on $L^p(\mu)$. (Hint: Use decreasing function $g(x) = (1+x)^p - x^p$.)

5. Let $h \to [0, \infty]$ be a measurable function, if $\mu(X) = 1$, set $A = \int_X h d\mu$, use Theorem 3.3 to prove that: $\sqrt{1 + A^2} \leq \int_X \sqrt{1 + h^2} d\mu \leq 1 + A$.

6. prove that μ is σ-finite if and only if there exists a positive function $f \in L^1(\mu)$.

7. If $f \in L^1(\mu) \cap L^\infty(\mu)$, then show that,

   (a) $f \in L^p(\mu)$ holds for every $1 < p < \infty$.

   (b) If μ is finite measure then $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

   (c) If μ is finite measure and if $1 < p < q \leq \infty$, then $L^q(\mu) \subset L^p(\mu)$.

8. Let L be a nonzero continuous linear functional on H and $M = \{x : Lx = 0\}$. Prove that $M^\perp$ is a vector space of dimension 1. (Hint: Use Theorem 4.11.)

9. Let $T : (C[0,1]; \|\cdot\|_\infty) \to (C[0,1]; \|\cdot\|_\infty)$ be a linear operator defined by $(Tf)(x) = x^3 f(x)$. Show that T is bounded and $\|I + T\| = 1 + \|T\|$. (Hint: Use function $f = 1$ to obtain that $\|T\| = 1$.)

10. If μ and ϕ are finite measures and $\mu(E) \leq \phi(E)$ holds for every measurable set $E \in \mathfrak{M}$ and if $f \in L^2(\phi)$, then, prove that there exists a unique measurable $g : X \to [0,1]$ with respect to ϕ such that $\int_X f d\mu = \int_X fg d\phi$. (Hint: Define linear functional $T(f) = \int_X f d\mu$ on Hilbert space $L^2(\phi)$ and use both theorems 1.40 and 4.12)

Good luck ♠